A PROP model for controllability and observability in linear time-invariant signal flow diagrams with feedback

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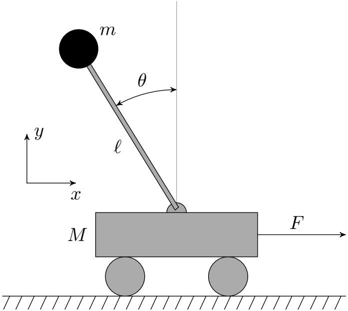
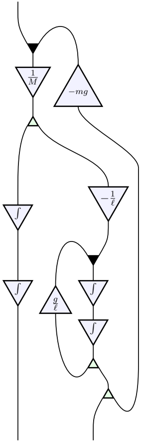
University of California, Riverside





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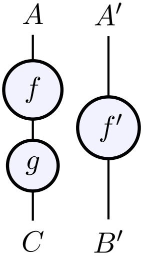
These signal flow diagrams look a lot like the string diagrams of morphisms in a PROP, so we will think of them as such. But what is a PROP? What is a string diagram?

PROPs

A **PROP** is a strict symmetric monoidal category with one object that generates all other objects by n-fold tensoring.

String diagrams depict morphisms in monoidal categories in a "two-dimensional" way:

Let $f\colon A \to B$, $f'\colon A' \to B'$, and $g\colon B \to C$. Then $(g \circ f) \otimes f'$ is



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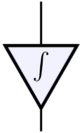
- `FinVect`, equivalent to the category of finite-dimensional vector spaces and linear maps
- `FinRel`, equivalent to the category of finite-dimensional vector spaces and linear *relations*
- `SigFlow`, equivalent to `FinRel` together with one extra "free" morphism from the base field to itself.

This extra morphism will be depicted 

A **linear relation** $F\colon U \nrightarrow V$ from a vector space U to a vector space V is a linear subspace $F \subseteq U \oplus V$.

When we compose linear relations $F\colon U \nrightarrow V$ and $G\colon V \nrightarrow W$, we get a linear relation $G \circ F\colon U \nrightarrow W$:

$$G \circ F = \{(u, w)\colon \exists v \in V \quad (u, v) \in F \text{ and } (v, w) \in G\}.$$

A linear map $\phi\colon U \to V$ gives a linear relation $F\colon U \nrightarrow V$, namely the graph of that map:

$$F = \{(u, \phi(u))\colon u \in U\}.$$

In this way, composing linear maps is a special case of composing linear relations.

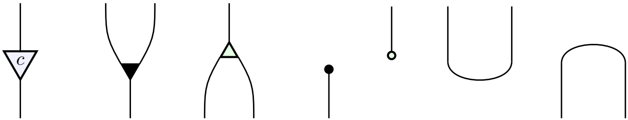
`FinVect` is a subPROP of `FinRel`.

Lemma (Baez, E.)

The category `FinRel`, *with*

- *finite dimensional vector spaces over* k *as objects,*
- *linear relations as morphisms,*

is a symmetric monoidal category with $\oplus$ *as its tensor product.* `FinRel` *is generated as a symmetric monoidal category by one object,* k, *together with the morphisms*



where $c \in k$.

Two systems of equations:

- Differential:
  $\dot{x} = Ax + Bu$
- Linear:
  $y = Cx + Du$

u is the input vector ($\in \mathbb{R}^m$)
x is the “state” vector ($\in \mathbb{R}^n$)
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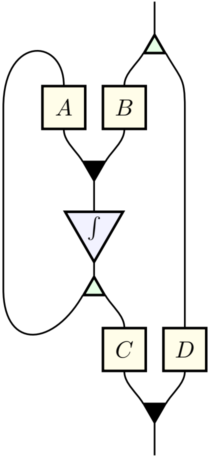

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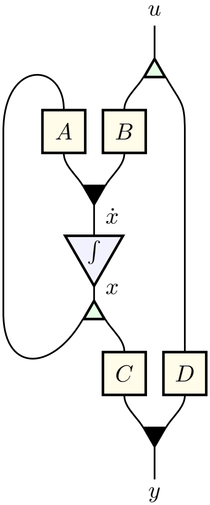

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Control engineers are often concerned with whether a system can be driven to a desired state using the inputs and whether the state can be determined from the system's outputs. These are the "dual" concepts of *controllability* and *observability*.

Controllability

Given our matrix equations

$$\dot{x} = Ax + Bu$$

and

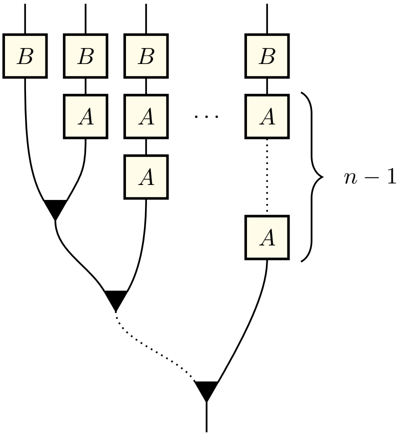
$$y = Cx + Du,$$

controllability is equivalent to the following matrix having full (row) rank:

$$\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}.$$

Controllability

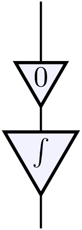
The equivalent description using signal flow diagrams says



is an onto map in `FinVect`.

An example

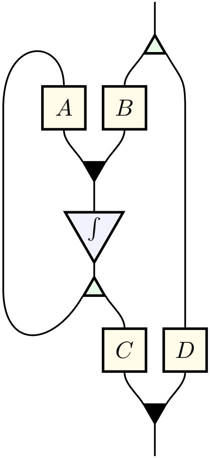
The signal flow diagram

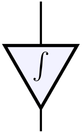


is not controllable, since the state vector cannot be influenced at all by the input.

Given an arbitrary signal flow diagram, can we find A, B, C, and D?

D



Replace 

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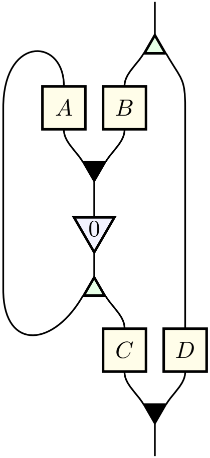

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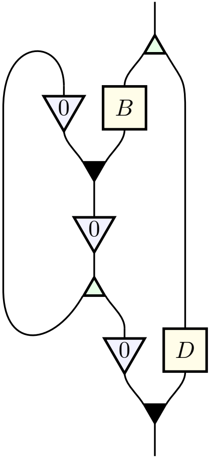

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B is discarded when it gets multiplied by zero, and all that's left is a single path from input through D to output.

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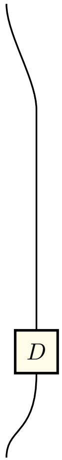

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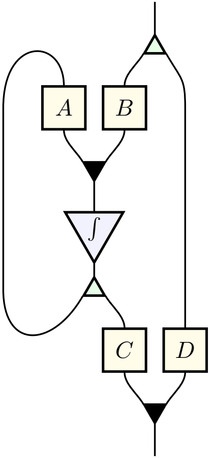
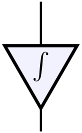
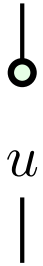


Replace  with 

and

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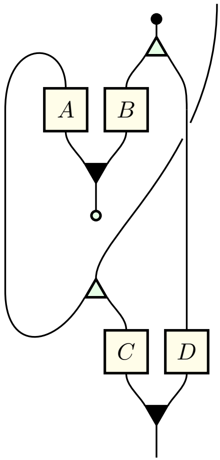
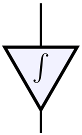


Replace  with 

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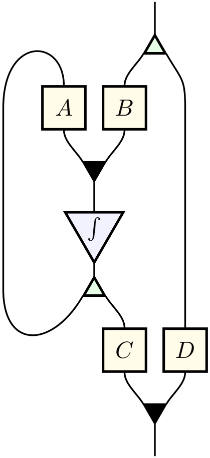
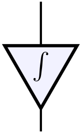


Replace 

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and

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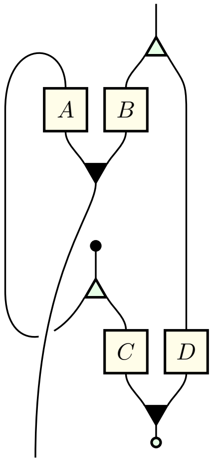
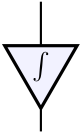
B



Replace  with 

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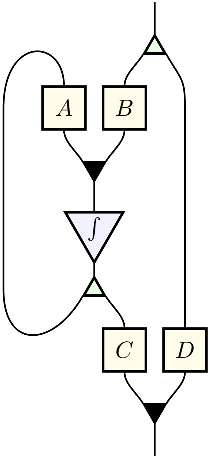
A



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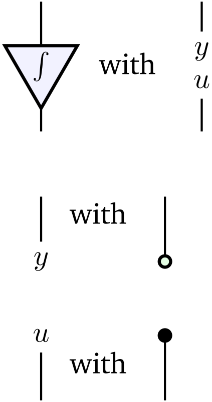

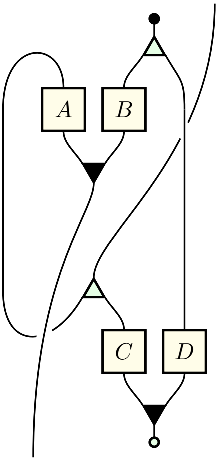

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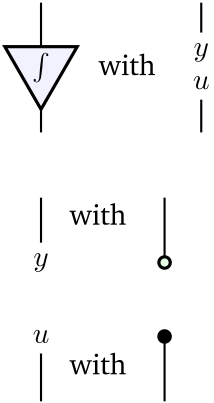
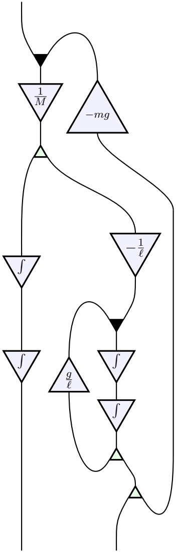


Example: Inverted pendulum

Let's apply this to the inverted pendulum:



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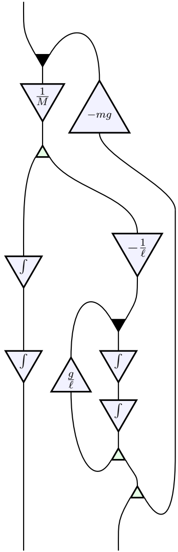
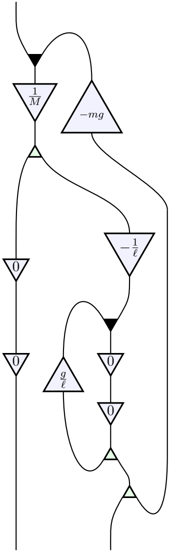


$D \quad =$



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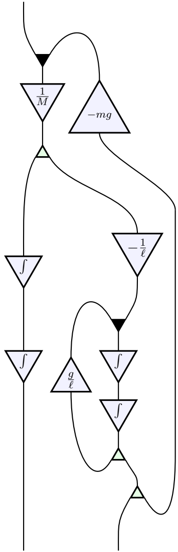
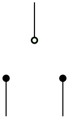


$D \quad =$ 

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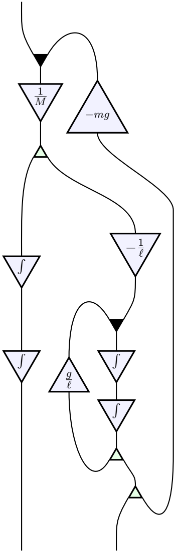
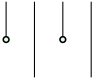


$C =$



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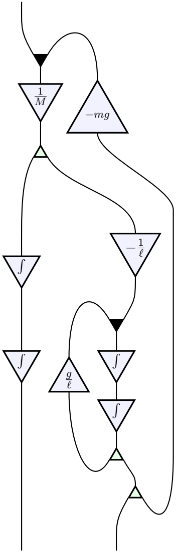

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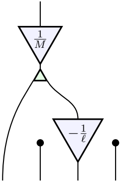

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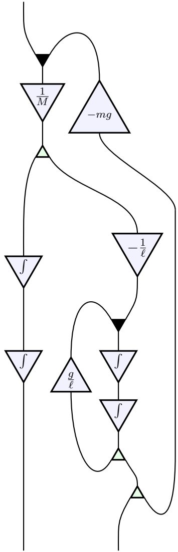

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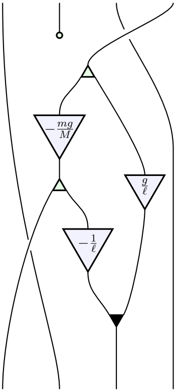

When we apply these to our controllability criterion, we get an onto map, which means the inverted pendulum is indeed controllable.

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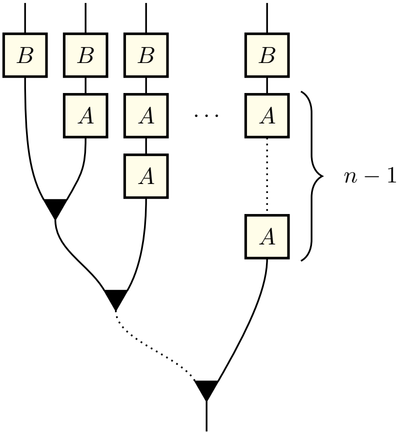
There is no reason why this process has to give us linear maps! A more general signal flow diagram will have linear relations for A, B, C, D. This leads to the following generalization.

Controllability, generalized

A signal flow diagram, decomposed into relations A, B, C, D, is controllable if



is an epimorphism in `FinRel`.

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Given our matrix equations

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observability is equivalent to the following matrix having full (column) rank:

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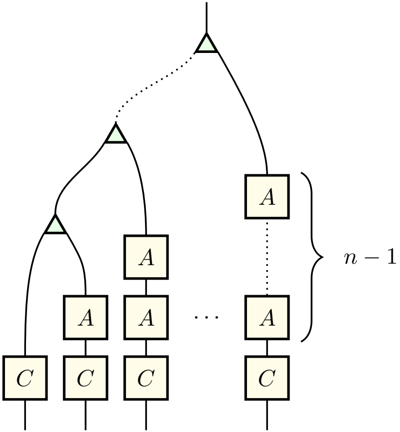

is a one-to-one map in `FinVect`

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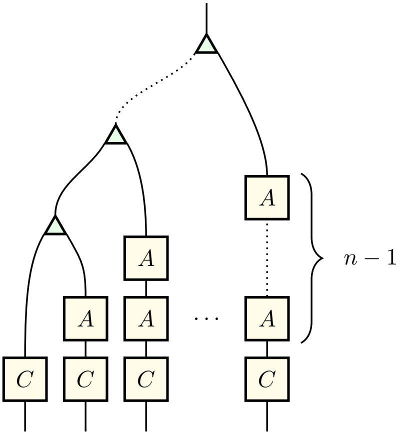

is a monomorphism in `FinRel`

“Duality”

The “duality” between controllability and observability, matrix version:

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This color swap means the dual of any linear map will still be a linear map.

Future work

Add time dependence and/or nonlinearity to this picture.

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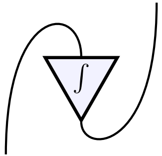
Extend to all signal flow diagrams, not just the ones that admit the A, B, C, D decomposition, such as



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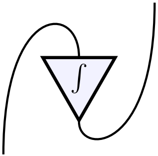

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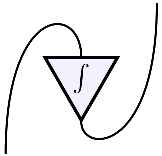

Compare and contrast the state-space approach here with the behavioral approach.

Consider other control theory topics, such as stability.

Thank you!

References

For more on “bizarro” duality, check out Paweł Sobociński’s blog, http://graphicallinearalgebra.net

For more on `FinRel` and `FinVect`, check out my article with John Baez, *Categories in control*